Announcements

1) Presentation, Stephen DeBacker, 4-5 Thursday, CB 2062, "The Seven Color Theorem"

2) HW #6 up later tonight or early tomorrow

Stability

Define

$$\rho = \frac{\max\limits_{\substack{1 \le i \le m \\ 1 \le j \le m}} |A(i,j)|}{\max\limits_{\substack{1 \le i \le m \\ 1 \le j \le m}} |U(i,j)|}$$

Theorem: Gaussian elimination with partial pivoting is backwards stable in the sense that $\exists\ \delta A \in \mathbb{C}^{m \times m}$,

$$\tilde{L}\tilde{U} = \tilde{P}A + \delta A$$

and

$$\frac{\|\delta A\|}{\|A\|} = O(\rho\, \varepsilon_{\text{machine}})$$

5×5 example in book:

$$p = 16 = 2^{5-1}.$$

In general, you can find

$$A \in \mathbb{C}^{m \times m}, \quad \rho(A) = 2^{m-1}.$$

Q: What about such examples?

A: All such example are contrived and do not occur in practice.

For "most" matrices, ρ is far less than exponential on the dimension.

Moral: Gaussian elimination with partial pivoting works just fine in practice.

Eigenvalues

Recall: (diagonalizability)

$A \in \mathbb{C}^{m \times m}$ is called

diagonalizable if there is

an invertible $S \in \mathbb{C}^{m \times m}$ and

diagonal $D \in \mathbb{C}^{m \times m}$,

$$\boxed{A = SDS^{-1}}$$

The eigenvalues of A are precisely the diagonal elements of D.

The product SDS^{-1} is called an eigenvalue decomposition for A.

Example 1: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A$

is not diagonalizable.

Why not?

$$A - \lambda I_2 = \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = \lambda^2 = 0$$

$\lambda = 0$ is the only eigenvalue of A.

If $A = SDS^{-1}$,

then the diagonal elements of D are the eigenvalues of A,

so $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

But then $SDS^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

which is not A!

So A is not diagonalizable

Definition: (eigenspace)

If $\lambda \in \mathbb{C}$ is an eigenvalue for $A \in \mathbb{C}^{m\times m}$, define the eigenspace E_λ to be the subspace of $\mathbb{C}^m$ given by

$$\boxed{E_\lambda = \{x \in \mathbb{C}^m \mid Ax = \lambda x\}}$$

Geometric and Algebraic Multiplicity

The geometric multiplicity of an eigenvalue λ is defined as $\dim(E_\lambda)$.

The algebraic multiplicity is the number of times the factor $x - \lambda$ occurs in the characteristic polynomial.

Theorem: (diagonalizability)

A matrix $A \in \mathbb{C}^{m \times m}$ is diagonalizable if and only if for all $\lambda \in \mathbb{C}$ such that λ is an eigenvalue of A, the geometric multiplicity of λ is equal to the algebraic multiplicity of λ.

Proof: $\Rightarrow$) Suppose A is diagonalizable, $A = SDS^{-1}$.

Then

$$\det(A) = \det(SDS^{-1})$$

$$= \det(S)\det(D)\det(S^{-1})$$

$$= \cancel{\det(S)}\det(D)\frac{1}{\cancel{\det(S)}}$$

$$= \det(D)$$

Since $A - \lambda I_m = SDS^{-1} - \lambda S I_m S^{-1}$
$= S(D - \lambda I_m)S^{-1}$,

we can the reduce to A

diagonal.

But here, the result is easy,

since if we write the

eigenvalues as $(\lambda_i)_{i=1}^{m}$

(with multiplicity), assuming

$\lambda_i = (i,i)$ entry of A,

e_i is an eigenvector for λ_i; the number of times λ_i repeats (which is the algebraic multiplicity) is then equal to the number of e_i's (basis for E_λ, total number $= \dim(E_\lambda)$).

Take math 413.

Example 2: (back to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$)

The only eigenvalue of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is zero, of algebraic multiplicity 2.

Now if

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so $y=0$ and

$$E_0 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \,\middle|\, x \in \mathbb{C} \right\}$$

$$= \left\{ x \begin{bmatrix} 1 \\ 0 \end{bmatrix} \,\middle|\, x \in \mathbb{C} \right\}.$$

Then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a

basis vector for E_0,

which implies

$\dim(E_0) = 1$

$< 2 =$ algebraic multiplicity.

Note: The algebraic multiplicity of an eigenvalue is always greater than or equal to the geometric multiplicity.

Nice examples of Diagonalizable Matrices

1) Diagonal matrices

2) Self-adjoint matrices (unitarily diagonalizable)

3) Unitary matrices

4) Normal matrices $(A^*A = AA^*)$

unitarily diagonal-izable

Schur Factorization

A Schur Factorization of $A \in \mathbb{C}^{m \times m}$ is a decomposition

$$\boxed{A = QTQ^*}$$

for some unitary $Q \in \mathbb{C}^{m \times m}$ and some upper triangular $T \in \mathbb{C}^{m \times m}$.

Theorem: Every matrix $A \in \mathbb{C}^{m \times m}$ has a Schur factorization.

Why would this be true?

First, an example.

Example 3: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

The Schur factorization of A is

$$A = I_2 A I_2$$

begin "proof"

2×2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

we know there is an eigenvalue λ for A with corresponding eigenvector v.

Want a unitary Q_1 with

$$Q_1 A Q_1^* = \begin{bmatrix} \lambda & e \\ 0 & f \end{bmatrix}$$

We pick the eigenvalue v to be of unit length, then let Q_1 be the unitary that maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to v.

Then $A = Q_1 \begin{bmatrix} \lambda & e \\ 0 & f \end{bmatrix} Q_1^*$.